

Effective potential (in)stability and lower bounds on the scalar (Higgs) mass

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Abstract

It is widely believed that the top loop corrections to the Higgs effective potential destabilise the electroweak (EW) vacuum and that, imposing stability, lower bounds on the Higgs mass can be derived. With the help of a scalar–Yukawa model, we show that this apparent instability is due to the extrapolation of the potential into a region where it is no longer valid. Stability turns out to be an intrinsic property of the theory (rather than an additional constraint to be imposed on it). However, lower bounds for the Higgs mass can still be derived with the help of a criterium dictated by the properties of the potential itself. If the scale of new physics lies in the TeV region, sizeable differences with the usual bounds are found. Finally, our results exclude the alternative meta-stability scenario, according to which we might be living in a sufficiently long lived meta-stable EW vacuum.

1 Introduction

The Standard Model (SM) of particle physics is a very successful theory which has received a great number of experimental confirmations. As is well known, however, it is not complete. Its scalar sector, in particular, poses deep (and so far unanswered) questions.

The value of the Higgs mass is not fixed by the theory, it is a free parameter. Nevertheless, in order to get informations on this fundamental quantity,

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theorists have tried to exploit at best the properties of the scalar sector of the SM (or some of its extensions).

Through the analysis of the scalar effective potential, upper and lower bounds on the Higgs mass, m_H , have been obtained as a function of the physical cutoff, the scale of new physics. The upper bounds come from the triviality of the quartic coupling [1] (for an alternative point of view see [2]), the lower ones from the requirement that the EW vacuum be stable (or, at least, meta-stable) [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

For the lower bounds, the analysis is performed with the help of the RG-improved effective potential, $V_{RGI}(\phi)$. Due to the $t\bar{t}$ loop corrections, V_{RGI} bends down for ϕ larger than v , the EW minimum. Depending on the value of the physical parameters, the resulting potential can be either unbounded from below up to the Plank scale, or can rise up again after forming a new minimum which is typically deeper than the EW vacuum. The latter is then said to be meta-stable.

As the instability occurs for sufficiently large values of the field, V_{RGI} is approximated by keeping only the quartic term [9]. Using standard notations:

$$V_{RGI}(\phi) \sim \frac{\bar{\lambda}(\phi)}{24} \phi^4. \quad (1)$$

In Eq.(1), the dependence of $\bar{\lambda}(\phi)$ on ϕ is essentially the same as that of the corresponding RG-improved quartic coupling constant, $\lambda(\mu)$, on the running scale μ , so that the behaviour of the effective potential can be read out from the $\lambda(\mu)$ flow ³.

The bending of the potential is due to the quarks-Higgs Yukawa couplings, namely to the minus sign carried by the fermion loops. Practically, it is sufficient to consider only the top, as the other (much lighter) quarks give comparably negligible contributions.

The physical request that the EW vacuum be stable against quantum fluctuation is seen as an *additional phenomenological constraint* to be imposed on the effective potential. This constraint induces a relation between the physical cutoff and the Higgs mass.

The derivation of the lower bounds goes as follows. Taking a boundary value for $\lambda(\mu)$ and for the other couplings, typically at $\mu = M_Z$, the coupled RG equation are runned. As μ increases, $\lambda(\mu)$ (initially) decreases. Depend-

³As correctly pointed out in [12], however, $\bar{\lambda}(\phi)$ contains also terms not contained in $\lambda(\mu)$. They are really negligible only for very large values of ϕ .

ing on its initial value, $\lambda(M_Z)$, it may happen that at a certain scale, $\mu = \Lambda$, the running coupling λ vanishes, becoming negative for higher values of μ . Requiring that the EW vacuum be stable, Λ is interpreted as the physical cutoff of the theory, the scale where new physics appears. From the matching condition, which relates m_H to $\lambda(M_Z)$ (at the tree level it is $m_H^2 = \frac{\lambda(M_Z)}{3} v^2$), a lower bound for m_H as a function of Λ is obtained. This is the stability bound.

The possibility of having a minimum deeper than the EW one is also considered. The argument is that, as far as the tunnelling time between the false (EW) and the true vacuum is sufficiently large compared to the age of the Universe, we may well be living in the meta-stable EW vacuum. In this case, meta-stability bounds on m_H are found [4, 6, 13].

These results, however, are at odds with a property of the effective potential, $V_{eff}(\phi)$, which, as is well known, is a convex function of its argument [14, 15, 16]. It is also known that, when the classical potential is not convex (the phenomenologically interesting case), at any finite order of the loop expansion, V_{eff} does not enjoy of this fundamental property. Alternative non-perturbative methods of computing the effective potential, though, such as lattice simulations [17], variational approaches [18], or suitable averages of the perturbative results [19], provide the proper convex shape. The Wilsonian RG approach also gives a non-perturbative convex approximation for V_{eff} [20, 21, 22, 23, 24].

One of the main goals of the present work is to show that V_{eff} is *nowhere unstable*. Its apparent instability is due to an extrapolation to values of ϕ which lie beyond its region of validity. Naively, however, the instability seems to occur in a region of ϕ where perturbation theory can be trusted [8] and this explains why previous analyses have missed this point ⁴.

We also show that, despite the convexity of the potential, actually thanks to this property, lower bounds for the Higgs mass can still be derived. Nevertheless, they no longer come as a result of an additional phenomenological constraint on V_{eff} , namely the requirement of stability, they are already encoded in the theory. As we shall see, if the scale of new physics lies in the TeV region, the difference between our bounds and those obtained with the help of the usual stability criterium becomes sizeable. The meta-stability

⁴In addition, the use of RG techniques, which enlarge the domain of validity of perturbation theory via the resummation of leading, next to leading, ... logarithms, leads to the believe that the derivation of this instability is theoretically sound [8].

scenario, on the contrary, is definitely excluded.

Finally, in order to shed more light on this (often mistreated and misunderstood) subject, we reconsider here some popular arguments [6, 25], sometimes quoted as the resolution of the instability (convexity) problem, and show that they are (at least) misleading. In section 2 we mainly concentrate on this last point which gives a good introduction to the subject and provides further motivation for our analysis.

To understand the origin of the instability, we do not need to consider the complete SM. The group and the gauge structure of the theory are not essential for its occurrence. As it is due to the top-Higgs coupling (actually to the minus sign carried by the $t\bar{t}$ -loop), the same instability occurs in the simpler model of a scalar coupled to a fermion with Yukawa coupling. To illustrate our argument, it will be sufficient to limit ourselves to consider this model. The extension of our results to the SM is immediate.

The instability of the scalar effective potential is the subject of many studies. The one-loop (or higher loops) and the RG-improved potential are computed with the help of dimensional regularization. We also begin by computing the effective potential of our model in the $\overline{\text{MS}}$ scheme (section 3). However, as will become clear in the following, dimensional regularization cannot reveal (in fact it masks) the origin of the problem.

The flaw in the usual procedure will be uncovered with the help of more physical renormalization schemes, the momentum cutoff regularization and the Wilsonian RG method. Dimensional regularization is a very powerful scheme which directly gives the finite results of renormalised perturbation theory. These other schemes allow to better follow the steps for the derivation of the renormalised potential from the bare one. This will help in finding the origin of the instability problem.

While completing our paper, we noted that this issue was recently considered in [26, 27]. Although our conclusions look similar to those reached by these authors, we believe that their work differs from our in some important aspects, worth to be discussed. A comparison will be presented in the conclusions.

The rest of the paper is organized as follows. In section 2 we show how the Bogolubov criterium of dynamical instability allows to reconcile the convexity of Γ_{eff} with the existence of a broken phase and how the broken phase Green's functions can be derived from (the convex) Γ_{eff} . Moreover, we show how the dynamical instability criterium can be implemented within the framework of the Wilsonian RG method. In section 3 we compute the

\overline{MS} one-loop and RG-improved effective potential for our model and see that they both are unstable. In section 4 the same problem is considered within the momentum cut-off regularization scheme. In section 5 we analyse the results of the previous section and show that the instability comes from an illegal extrapolation of the renormalised potential beyond its range of validity. In addition, consistently with the stability constraint, we consider a criterium for finding the physical cutoff of the theory. In section 6 we apply this criterium to the SM, thus getting lower bounds on the Higgs mass as a function of the scale of new physics, and compare with previous results. In section 7 we reconsider the instability problem within the framework of the non-perturbative Wilsonian RG method. Section 8 is for the summary and for our conclusions.

2 Broken phase and dynamical instability.

Before starting the detailed study of our model, in the present section we carefully analyse some popular arguments [6, 25], often presented as the resolution of the instability problem, and show that they are misleading. Moreover, by combining the Bogolubov criterium of dynamical instability with the Wilsonian RG method, we shall provide further support to our analysis.

In [6, 25] the effective action, $\Gamma_{eff}[\phi]$, and the generating functional of the broken phase 1PI vertex functions, $\Gamma_{1PI}[\phi]$, are presented as two different functionals. Actually, these authors consider the first order in the \hbar -expansion of Γ_{1PI} , Γ_{1PI}^I , and note that it is not convex. It is then argued that, when studying the stability of the EW vacuum, the relevant quantity to consider is V_{1PI} (or, more generally, its RG-improved version, V_{RGI}) rather than the convex V_{eff} , and that, being V_{1PI} non-convex, there is no convexity (instability) problem [6]⁵.

The argument is the following. $V_{eff}(\phi)$ comes from the minimisation of $\langle \psi | \hat{H} | \psi \rangle$, where \hat{H} is the energy density of the system and $|\psi\rangle$ is a state

⁵Presenting $\Gamma_{1PI}[\phi]$ and $\Gamma_{eff}[\phi]$ as two different quantities is a first source of confusion. As we have already said, the convexity property of the exact Γ_{eff} cannot be recovered within the loop expansion. Γ_{1PI}^I , which is the quantity considered in [6, 25], is a non-convex, $O(\hbar)$, approximation of Γ_{eff} . It correctly approximates Γ_{eff} in the neighbourhood of the minima (with some warnings specified later). In the region where it is non-convex, however, it is a bad approximation of Γ_{eff} .

which satisfies the constraint $\langle \psi | \hat{\phi} | \psi \rangle = \phi$. For a symmetry breaking classical potential, the states that correspond to values of ϕ in the region between the classical minima, are not localised (more on this point later). As only localised states are of interest to us, and V_{1PI}^{1l} is supposed to correspond to localised states also in the region between the minima [19], the conclusion is that V_{1PI}^{1l} rather than V_{eff} is the appropriate potential to consider.

It is not difficult to see, however, that these lines of reasoning are misleading. Indeed, the instability occurs for values of ϕ above v . Now, differently from those related to the region $-v \leq \phi \leq v$, the states that correspond to this range of ϕ are perfectly well localised and the above argument does not apply.

Moreover, as we shall briefly show below, the broken phase zero momentum Green's functions, $\Gamma_n^{(v)}$, can be obtained from the convex V_{eff} once we consider a physical procedure [28, 29] based on the dynamical instability of the classical vacua (Bogolubov criterium) and that the usual loop expansion for V_{eff} can be obtained within this framework.

This will help to further clarify the relation between V_{eff} and V_{1PI}^{1l} . In any case, the potential to consider is V_{eff} , which is everywhere convex. However, as long as we are only interested in the broken theory Green's functions, i.e. in the local properties of V_{eff} at $\phi = v$, it is possible (and from a practical point of view even more convenient) to consider a non-convex approximation, as V_{1PI}^{1l} (or higher order ones), which coincides with V_{eff} in the neighbourhood of v (see below and footnote 5).

Actually, the only range of ϕ 's where a significative difference between the loop approximation and the exact effective potential is expected is the internal region, $-v \leq \phi \leq v$. The reason is easy to understand. By construction, the one-loop approximation for the path integral which defines the effective potential considers the expansion of the action around a single saddle point. For values of ϕ in the internal region, however, there are two competing saddle points having the same weight [16]. Taking into account both of these contributions, we get for the effective potential the known flat (convex) shape between the classical minima (Maxwell construction). On the contrary, for $\phi \geq v$ the path integral is dominated by a single saddle point. Therefore, no significative difference can occur in this region between the one-loop (or higher loop) approximation and the exact effective potential.

A similar argument can be given within the framework of the Wilsonian RG approach where it was shown that, differently from the unbroken phase, the path integral which defines the infinitesimal RG-transformation for the

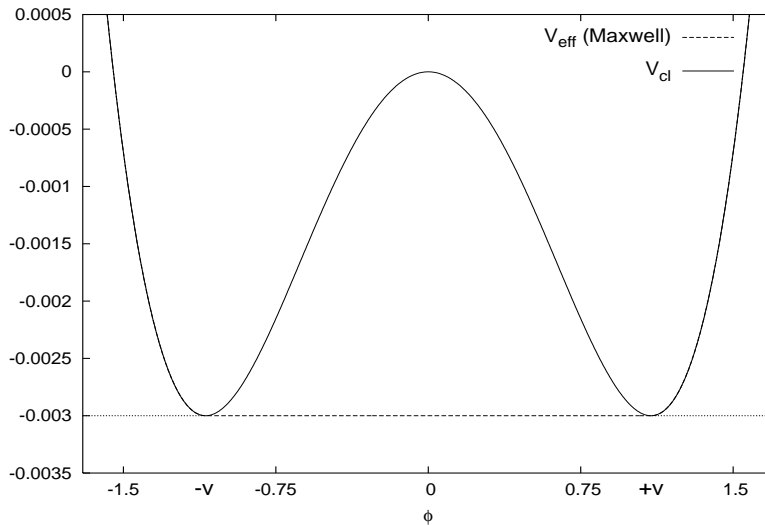


Figure 1: The Maxwell construction for the classical potential of the single component scalar theory considered in the text. The parameters are chosen as: $\lambda = 5 \cdot 10^{-2}$ and $m^2 = -10^{-2}$.

Wilsonian potential in the broken phase is saturated by non-trivial saddle points [23].

Now we briefly show how the $\Gamma_n^{(v)}$'s are obtained from the convex effective action Γ_{eff} . For illustrational purposes, it is sufficient to consider the case of a constant background field, i.e. to consider V_{eff} rather than the full effective action. Anyway, in the following, we are only interested in V_{eff} . For the sake of simplicity, we also limit ourselves to the case of a single component scalar theory.

General theorems [14, 30], together with several analytical and numerical non-perturbative studies [17, 18, 19, 20, 21, 22, 23, 24], indicate that V_{eff} is a convex function of ϕ with a flat bottom between $-v$ and v , the minima of the classical potential. At the lowest order, V_{eff} coincides with the well known Maxwell (or double tangent) construction sketched in fig.1.

The (zero momentum) $\Gamma_n^{(v)}$'s should be obtained by taking the derivatives of V_{eff} at $\phi = v$. Due to the shape of the potential, however, this operation is ambiguous and has to be defined with a certain care.

The approach that we are going to consider now [28, 29], far from being a technical point, has a deep physical meaning. Following Bogolubov, in

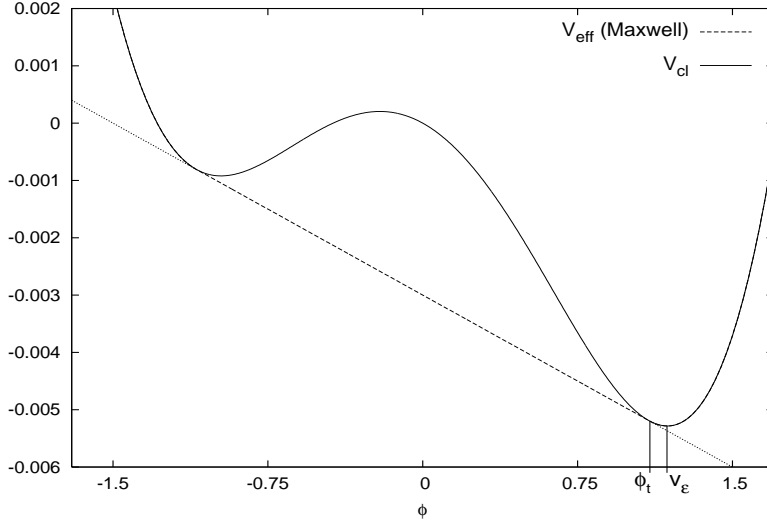


Figure 2: The Maxwell construction for the classical potential of the single component scalar theory with an explicit symmetry breaking term, $-\varepsilon\phi$. The parameters are chosen as: $\lambda = 5 \cdot 10^{-2}$, $m^2 = -10^{-2}$ and $\varepsilon = 2 \cdot 10^{-3}$.

fact, we interpret the occurrence of symmetry breaking as a manifestation of the “dynamical instability” of the otherwise equivalent vacua of the potential. Adding to the Lagrangian an infinitesimal source term which explicitly breaks the classical symmetry of the theory, $-\varepsilon\phi$, we select one of the two classical vacua (see fig.2). More precisely, this additional term creates an absolute minimum, v_ε , close to the old v .

As for the symmetric case, the lowest order for V_{eff} can be obtained with the help of the double tangent construction (fig.2). A simple inspection of fig.2 shows that the derivatives at $\phi = v_\varepsilon$ of the resulting modified effective potential, $V_{eff}(\phi; \varepsilon)$, can be safely taken. In fact, while in the symmetric case (fig.1) the flat region extends from one of the classical minima to the other (the minima coincide with the tangent points), in fig.2 the effective potential (as the classical one) has an absolute minimum, v_ε , and the flat region starts at $\phi_t < v_\varepsilon$. The corresponding $\Gamma_n^{(v_\varepsilon; \varepsilon)}$'s at this order are then easily obtained. The successive $\varepsilon \rightarrow 0$ limit ⁶ gives the desired $\Gamma_n^{(v)}$'s.

⁶Although in this brief presentation we do not aim at complete rigour, it is worth to point out that to construct the $\Gamma_n^{(v)}$'s we begin first with a finite volume system and successively take the infinite volume limit. The latter has to be taken previous to the

Clearly, the $\Gamma_n^{(v)}$'s that we get this way are nothing but the usual tree level $\Gamma_n^{(v)}$'s. To get higher order approximations, we need to go beyond this lowest order Maxwell construction. Following [23], we now show that, with the help of the Wilsonian RG approach, the above results can be established beyond this order.

As is well known, the non-perturbative RG equation for the Wilsonian effective potential, $U_k(\phi)$, in $d = 4$ dimensions can be written as [31, 32, 33]:

$$k \frac{\partial}{\partial k} U_k(\phi) = -\frac{k^4}{16\pi^2} \ln \left(\frac{k^2 + U_k''(\phi)}{k^2 + U_k''(0)} \right), \quad (2)$$

where the prime indicates derivation w.r.t. ϕ . Note that the classical (bare) potential is $V_{cl}(\phi) = U_\Lambda(\phi)$, while the effective potential is $V_{eff}(\phi) = U_{k=0}(\phi)$.

For a theory in the broken phase, however, Eq.(2) becomes unstable. More precisely, for values of ϕ in the internal region, this equation develops a singularity at finite critical values, $k_{cr}(\phi)$, of the running scale k . Starting from $k = k_{cr}(\phi)$, Eq.(2) is no longer valid.

In [23] a new non-perturbative RG equation for ϕ in the unstable region was established:

$$U_{k-\delta k}(\phi) = \min_{\{\varrho\}} \left[k^2 \varrho^2 + \frac{1}{2} \int_{-1}^1 dx U_k(\phi + 2\varrho \cos(\pi x)) \right]. \quad (3)$$

The minimum of Eq.(3), $\varrho_k(\phi)$, is the amplitude of the non-trivial saddle point which dominates the path integral defining the infinitesimal RG-transformation ($k \rightarrow k - \delta k$) in the internal region. In the external region, on the contrary, the path integral is dominated by the trivial saddle point, i.e. $\varrho_k(\phi)$ vanishes.

In [23] the case of the symmetric potential (fig.1) was considered and the Maxwell construction for V_{eff} was established. Here we extend this analysis to the case of the potential with an explicit symmetry breaking term.

In fig.3 we show the flow of the Wilsonian potential, $U_k^{(\varepsilon)}(\phi)$, starting from the critical values $k_{cr}(\phi)$. From this figure we see that, even in the asymmetric case, there is a region where the effective potential, $V_{eff}^{(\varepsilon)}(\phi) = U_{k=0}^{(\varepsilon)}(\phi)$, is flat and coincides with the double tangent construction. The same considerations done for the lowest order result are valid. In particular, the tangent point is displaced to the left of v_ε and the derivatives of $V_{eff}^{(\varepsilon)}(\phi)$ at v_ε can be safely taken.

$\varepsilon \rightarrow 0$ limit.

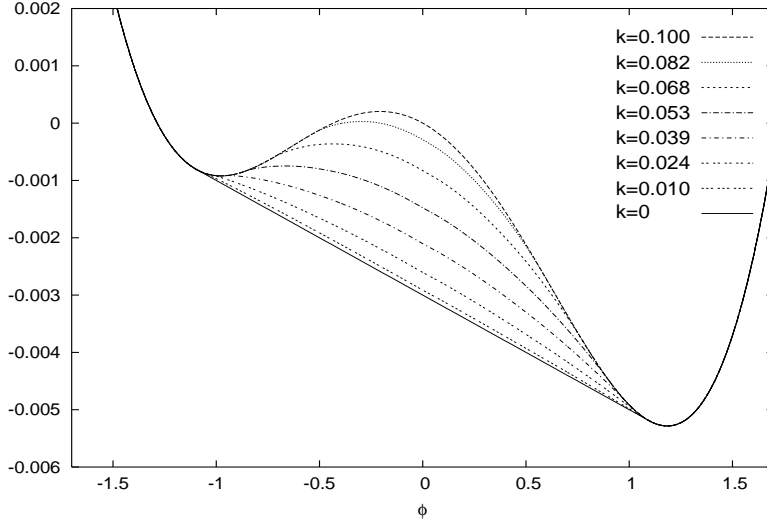


Figure 3: RG flow for the potential of the single component scalar theory with explicit symmetry breaking term. Only the flow in the internal region is considered, i.e. the flow given by Eq.(3). The boundary values for the parameters at $k = 0.1$ are: $\lambda = 5 \cdot 10^{-2}$, $m^2 = -10^{-2}$ and $\varepsilon = 2 \cdot 10^{-3}$.

The general conclusion of this analysis is that, with the help of Eqs.(2) and (3), the Wilsonian potential can be runned all the way down from $k = \Lambda$ to $k = 0$. The result is a non-perturbative convex approximation for V_{eff} which shows the typical flat shape in the internal region (given by the running of Eq.(3)), while in the external region has the shape governed by Eq.(2).

We consider now the one-loop potential, $V^{1l}(\phi; \varepsilon)$. In view of the previous discussion, it is not difficult to understand that, as far as we limit ourselves to consider a range of values of ϕ sufficiently close to the absolute minimum, $V^{1l}(\phi; \varepsilon)$ provides a good approximation for $V_{eff}(\phi; \varepsilon)$. Clearly, this is true for higher order loops too.

Before ending this section, we would like to expand, as anticipated, on the argument according to which, when studying the stability of the vacuum, the convex V_{eff} is not the appropriate potential to consider [6].

Let us indicate with $|v\rangle$ and $|-v\rangle$ the vacua constructed around $\phi = v$ and $\phi = -v$ respectively. The flatness of V_{eff} in the $-v < \phi < v$ region implies that all the linear combinations of states $\alpha|v\rangle + \beta|-v\rangle$ (with $|\alpha|^2 + |\beta|^2 = 1$) are equivalent vacua, they all have the same energy. Apart from the trivial

ones ($|\alpha| = 1$, $\beta = 0$ and $\alpha = 0$, $|\beta| = 1$), with any of the other non trivial combinations we would obtain Green's functions which violate the cluster decomposition property. Moreover, the expectation value of the field is not constant allover V , the quantisation volume. In fact, for the generic state $\alpha|v\rangle + \beta|-v\rangle$, the expectation value $\langle\phi\rangle$ is given by $(|\alpha|^2 - |\beta|^2)v$, and V contains a fraction $|\alpha|^2$ of $\langle\phi\rangle = v$ and a fraction $|\beta|^2$ of $\langle\phi\rangle = -v$. Clearly, these states are not localised.

The above considerations are viewed as an indication that the convex V_{eff} is not the appropriate potential to deal with. Although correct, these observations have nothing to do with the instability problem. As we have just seen, the non localised states correspond to values of ϕ in the internal region. The instability problem, however, occurs in the external region, where the states are perfectly well localised. Moreover, with the help of the Bogolubov criterium, we have seen how the degeneracy in the internal region is lifted and (in the infinite volume limit) only one vacuum is selected.

3 One-loop and RGI potential. \overline{MS} Scheme.

We compute now the one-loop effective potential, V^{1l} , for our model in the \overline{MS} scheme and the corresponding RG-improved potential, V_{RGI} .

The model consists of a single scalar field plus a single fermion field with scalar quartic interaction and Yukawa coupling, i.e.:

$$\mathcal{L}(\phi, \psi, \bar{\psi}) = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + i \bar{\psi} \gamma_\mu \partial_\mu \psi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{2} \phi^4 + g \phi \bar{\psi} \psi \right). \quad (4)$$

Straightforward application of the \overline{MS} prescriptions gives:

$$\begin{aligned} V^{1l}(\phi) = & \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 + \frac{1}{64\pi^2} \left(m^2 + \frac{\lambda}{2} \phi^2 \right)^2 \left(\ln \left(\frac{m^2 + \frac{\lambda}{2} \phi^2}{\mu^2} \right) - \frac{3}{2} \right) \\ & - \frac{g^4 \phi^4}{16\pi^2} \left(\ln \frac{g^2 \phi^2}{\mu^2} - \frac{3}{2} \right), \end{aligned} \quad (5)$$

where m^2 , λ and g depend on the renormalization scale μ :

$$m^2 = m^2(\mu), \quad \lambda = \lambda(\mu), \quad g = g(\mu). \quad (6)$$

In the r.h.s. of Eq.(5), the fermionic contribution comes with a negative sign. Therefore, we can easily find values of λ and g (with $g^4 > \lambda$), together with a range of values of ϕ , which satisfy the perturbative conditions,

$$\lambda < 1, \quad g < 1 \quad \text{and} \quad \frac{g^4}{16\pi^2} \ln \frac{g^2 \phi^2}{\mu^2} < 1, \quad (7)$$

so that $V^{1l}(\phi)$ bends down and becomes lower than $V^{1l}(v)$ (see fig.4). This is the instability problem for our one-loop potential.

As is well known, we can improve on this result with the help of renormalization group techniques. Let us consider the one-loop RG functions for λ , g , m^2 and for the vacuum energy ⁷ Ω :

$$\begin{aligned} \beta_\lambda &= \frac{3\lambda^2}{16\pi^2} - \frac{3g^4}{\pi^2} & ; & & \beta_g &= \frac{g^3}{8\pi^2} \\ \beta_\Omega &= \frac{\lambda m^4}{32\pi^2} & ; & & \gamma_{m^2} &= \frac{\lambda}{16\pi^2}. \end{aligned} \quad (8)$$

The largest logarithmic correction in the r.h.s. of Eq.(5) comes from the last term (the fermion). According with the RG-improvement logic, we now choose the running variable t so that we get rid of this term in the improved potential: $t = \frac{1}{2} \ln \frac{g^2 \phi^2}{\mu^2} - \frac{3}{4}$. As usual, the running functions $\bar{\lambda}(t)$, $\bar{g}(t)$, $\bar{m}^2(t)$, and $\bar{\Omega}(t)$ are defined as the solutions of the differential equations:

$$\begin{aligned} \frac{d\bar{\lambda}}{dt} &= \beta_\lambda(\bar{\lambda}, \bar{g}, \bar{\Omega}, \bar{m}^2) & ; & & \frac{d\bar{g}}{dt} &= \beta_g(\bar{\lambda}, \bar{g}, \bar{\Omega}, \bar{m}^2) \\ \frac{d\bar{m}^2}{dt} &= \gamma_{m^2}(\bar{\lambda}, \bar{g}, \bar{\Omega}, \bar{m}^2) & ; & & \frac{d\bar{\Omega}}{dt} &= \beta_\Omega(\bar{\lambda}, \bar{g}, \bar{\Omega}, \bar{m}^2) \end{aligned} \quad (9)$$

with boundary conditions:

$$\bar{\lambda}(t=0) = \lambda; \quad \bar{g}(t=0) = g; \quad \bar{\Omega}(t=0) = 0; \quad \bar{m}^2(t=0) = m^2. \quad (10)$$

It is not difficult to see that the differential equations (9) can be solved analytically. For $\bar{g}(t)$ and $\bar{\lambda}(t)$, for instance, we have:

$$\bar{g}(t) = g \left(1 - \frac{g^2 t}{4\pi^2} \right)^{-\frac{1}{2}}$$

⁷When considering the RG-improvement, the cosmological constant term has to be taken into account even if it was originally absent.

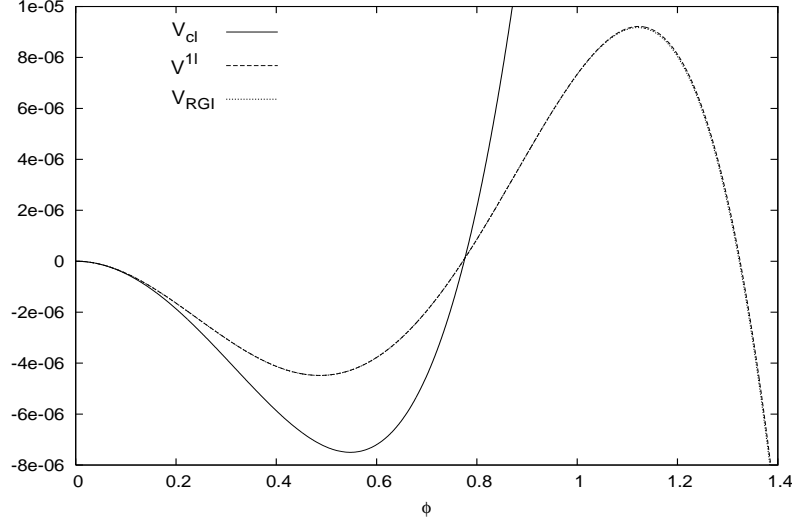


Figure 4: Together with the classical potential, V_{cl} , of Eq.(4), here we plot the one-loop, V^{1l} , and the RG-improved, V_{RGI} , effective potential. The parameters are chosen at the scale $\mu = 1.1 \cdot 10^{-1}$ and are: $\lambda = 2 \cdot 10^{-3}$, $m^2 = -10^{-4}$, $g = 3 \cdot 10^{-1}$. The instability of V^{1l} and V_{RGI} is immediately evident. Moreover, in this region, they are very close.

$$\bar{\lambda}(t) = \frac{2}{3}\bar{g}^2(t) \left(1 - \alpha + 2\alpha \left[1 + \left(\frac{\bar{g}(t)^2}{g^2} \right)^\alpha \frac{2g^2(\alpha + 1) - 3\lambda}{2g^2(\alpha - 1) + 3\lambda} \right]^{-1} \right), \quad (11)$$

with $\alpha = \sqrt{37}$.

Finally, the one-loop RG-improved potential is:

$$V_{RGI} = \frac{1}{2}\bar{m}^2(t)\phi^2 + \frac{\bar{\lambda}(t)}{24}\phi^4 + \bar{\Omega}(t) + \left(\frac{\bar{m}^2(t) + \frac{\bar{\lambda}(t)}{2}\phi^2}{64\pi^2} \right)^2 \ln \frac{\bar{m}^2(t) + \frac{\bar{\lambda}(t)}{2}\phi^2}{\bar{g}^2(t)\phi^2}. \quad (12)$$

In fig. 4 we plot V_{RGI} together with the one-loop and the classical potential for a particular choice of the renormalised parameters. A simple inspection of this figure shows that V_{RGI} (as well as V^{1l}) is unstable.

Before ending this section, we would like to note that, due to the competition between the λ^2 and the g^4 terms in β_λ (first of Eqs.(8)), $\lambda(\mu)$, after decreasing for a certain range of energy, finally increases (toward the Landau

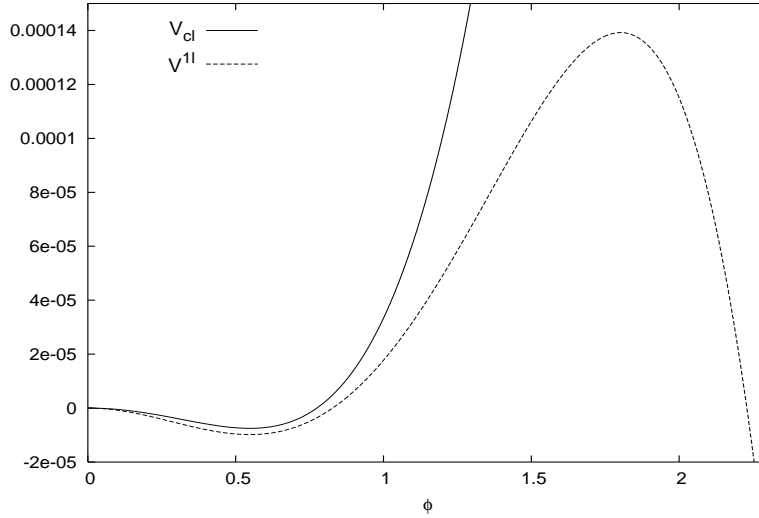


Figure 5: Differently from Fig.4, here we have implemented the RG conditions so that the location of the minimum and the curvature of V^{ll} at this point are the same as for V_{cl} (see Appendix B). The parameters are chosen as in Fig.4.

pole). This generates a second minimum in the effective potential, typically lower than the first one.

Now, for certain values of m_t and m_H , which are compatible with the current experimental determinations and limits, the Higgs effective potential of the SM shows such a behaviour already below the Planck scale. As the tunnelling time between the false (EW) and the true vacuum appears to be sufficiently large (as compared to the age of the Universe), the alternative scenario of a meta-stable EW vacuum is also considered and lower meta-stability bounds on the Higgs mass are derived [6, 13]. As we have anticipated, however, the proper treatment of the problem will show that effective potential is nowhere unstable. As a consequence, this scenario will be excluded.

4 Momentum cutoff scheme

In this section we show how the one-loop renormalised effective potential of Eq.(5) is obtained by considering the theory defined with a momentum cutoff. To prepare the discussion of the next section, we follow the computation in some detail.

The parameters of the Lagrangian are now the bare ones. Therefore, in Eq.(4) we replace m^2 , λ and g with m_Λ^2 , λ_Λ and g_Λ respectively. As in the previous section, for the sake of simplicity, we neglect the wave function renormalization ⁸. A straightforward application of perturbation theory gives:

$$\begin{aligned} V^{1l}(\phi) = & \frac{m_\Lambda^2}{2}\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 + \frac{1}{64\pi^2} \left\{ \Lambda^4 \ln \left(\frac{\Lambda^2 + m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}{\Lambda^2} \right) \right. \\ & + \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2 \right) \Lambda^2 - \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2 \right)^2 \ln \left(\frac{\Lambda^2 + m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}{m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2} \right) \Big\} \\ & - \frac{1}{16\pi^2} \left\{ \Lambda^4 \ln \left(1 + \frac{g_\Lambda^2\phi^2}{\Lambda^2} \right) + g_\Lambda^2\phi^2\Lambda^2 - g_\Lambda^4\phi^4 \ln \left(\frac{\Lambda^2 + g_\Lambda^2\phi^2}{g_\Lambda^2\phi^2} \right) \right\}. \end{aligned} \quad (13)$$

Considering only values of ϕ small compared to the cutoff,

$$\frac{\phi}{\Lambda} < 1, \quad (14)$$

expanding the r.h.s. of Eq.(13) in powers of $\frac{\phi}{\Lambda}$ and neglecting terms which are suppressed by negative powers of Λ , we get :

$$\begin{aligned} V^{1l}(\phi) = & \frac{m_\Lambda^2}{2}\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 - \frac{1}{16\pi^2} \left\{ 2g_\Lambda^2\phi^2\Lambda^2 - g_\Lambda^4\phi^4 \left[\ln \left(\frac{\Lambda^2}{g_\Lambda^2\phi^2} \right) + \frac{1}{2} \right] \right\} \\ & + \frac{1}{64\pi^2} \left\{ 2 \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2 \right) \Lambda^2 - \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2 \right)^2 \left[\ln \left(\frac{\Lambda^2}{m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2} \right) + \frac{1}{2} \right] \right\}. \end{aligned} \quad (15)$$

We now move from bare to renormalised perturbation theory. After performing the splitting of the bare parameters in the usual way:

⁸When, in section 6, we shall be interested in the derivation of lower bounds on the Higgs mass, the anomalous dimension will be appropriately taken into account.

$$m_\Lambda^2 = m^2 + \delta m^2, \quad \lambda_\Lambda = \lambda + \delta\lambda, \quad g_\Lambda = g + \delta g, \quad (16)$$

we insert Eq.(16) in Eq.(15) neglecting the higher order terms, i.e. removing δm^2 , $\delta\lambda$ and δg from the quantum fluctuation contribution. Finally, the counter-terms are determined so to cancel the quadratic and logarithmic divergences of V^{1l} .

There is an arbitrariness in the determination of the counter-terms (different renormalization conditions) which is reflected in an arbitrariness in the finite parameters of the renormalised potential. By choosing:

$$\begin{aligned} \delta m^2 &= \delta m_{bos}^2 + \delta m_{fer}^2 \\ \delta\lambda &= \delta\lambda_{bos} + \delta\lambda_{fer} \end{aligned} \quad (17)$$

with (μ is an arbitrary low energy scale)

$$\begin{aligned} \delta m_{bos}^2 &= -\frac{\lambda\Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2} \left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1 \right] \quad ; \quad \delta m_{fer}^2 = \frac{g^2\Lambda^2}{4\pi^2} \\ \delta\lambda_{bos} &= \frac{3\lambda^2}{32\pi^2} \left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1 \right] \quad ; \quad \delta\lambda_{fer} = -\frac{3g^4}{2\pi^2} \left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1 \right], \end{aligned} \quad (18)$$

we get:

$$\begin{aligned} V^{1l}(\phi) &= \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{64\pi^2} \left(m^2 + \frac{\lambda}{2}\phi^2 \right)^2 \left(\ln\left(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2}\right) - \frac{3}{2} \right) \\ &\quad - \frac{g^4\phi^4}{16\pi^2} \left(\ln\frac{g^2\phi^2}{\mu^2} - \frac{3}{2} \right), \end{aligned} \quad (19)$$

that is the one-loop potential of Eq.(5).

As for Eq.(5), the renormalised parameters that appear in Eq.(19) are defined at the scale μ . Now, repeating the same steps of the previous section, we obtain from Eq.(19) the RG-improved potential of Eq.(12).

5 Stability of the Effective Potential

We show now that the effective potential is nowhere unstable, the claimed (apparent) instability being due to the extrapolation of V^{1l} (V_{RGI}) into a region where it is no longer valid.

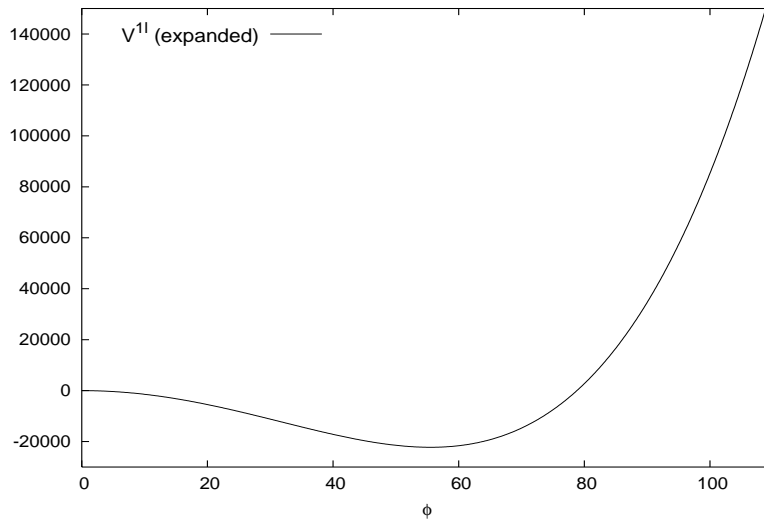


Figure 6: The one-loop effective potential of Eq.(15) (before the subtraction of the quadratic divergences), for $\lambda_\Lambda = 5 \cdot 10^{-2}$, $m_\Lambda^2 = -10^{-2}$, $g_\Lambda = 0.35$ and $\Lambda = 100$. Neglecting, as explained in the text, the internal region, we see that beyond the minimum the potential is convex.

Before turning our attention to the renormalised potential, we begin by considering the bare theory as defined by the one-loop potential of Eq.(15). For a certain region in the $(m_\Lambda^2, \lambda_\Lambda, g_\Lambda)$ -parameter space, this potential, as the classical one, has two minima (Higgs phase). As we have already explained, the loop expansion is inadequate for the region between the minima. In the following, we ignore this region and concentrate our attention only on the external one, where the loop-expansion is expected to hold (as we know, in the internal region the convexity is restored via the Maxwell construction).

A careful analysis of Eq.(15) shows that, in the external region, and within the range of ϕ where the potential is defined, i.e. for $\frac{|\phi|}{\Lambda} < 1$, the bare effective potential is convex (in agreement with exact theorems). Therefore, it does not present any instability. In fig.6 we show a plot of V^{1l} , Eq.(15), for a particular choice of the parameters.

We now subtract from the bare potential of Eq.(15) the quadratically divergent terms ⁹. As illustrated in fig.7 for a specific choice of the param-

⁹As is well known, a well defined physical meaning can be attached to this operation. In a non-supersymmetric scenario, this cancellation is interpreted as the result of

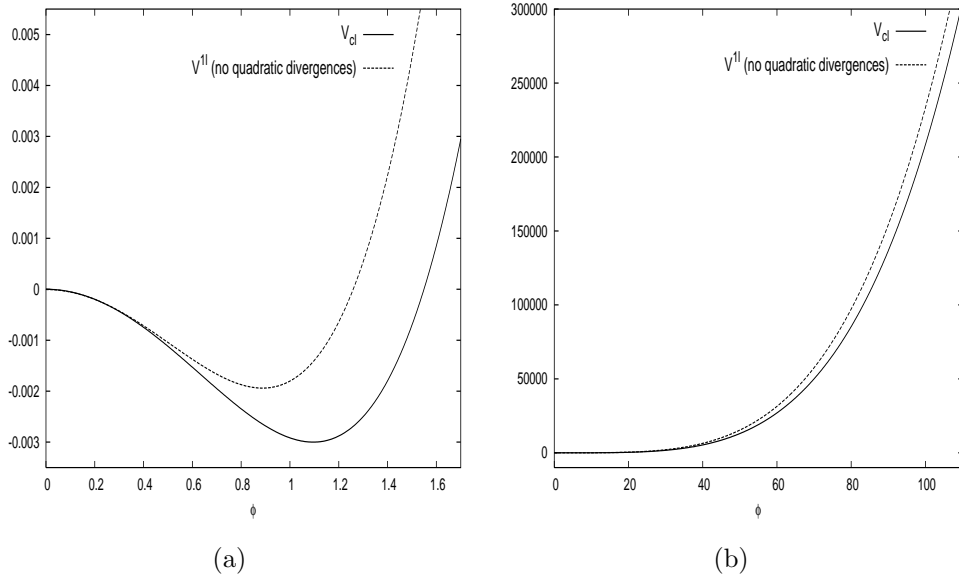


Figure 7: The bare potential together with the one-loop potential after subtraction of the quadratic divergences. The bare parameters are chosen as in Fig.5. (a) We zoom on a small region of ϕ , close to the classical minimum. (b) Here we see that the effective potential, up to the cutoff scale, is stable.

eters, again the resulting potential turns out to be convex (once more, only the external region has to be considered). The bare potential, even after the subtraction of the quadratic divergences, does not show any sign of instability.

As an aside remark, we note that, as they describe different degrees of freedom, the potentials of figs.6 and 7 actually belong to two different effective theories (with or without the Λ^2 terms). From the point of view of the phenomenological applications in particle physics, however, we are typ-

the conspiracy between unknown degrees of freedom, which live above the cutoff, and the quantum fluctuations of the fields below the cutoff. This way, the scalar (Higgs) mass is protected from getting too large corrections from the quantum fluctuations (this interpretation, however, poses the problem of the fine tuning required for the cancellation, the naturalness problem). In a susy scenario, on the contrary, this cancellation is obtained in a more “natural” way. It is due to the presence of additional degrees of freedom (fields) below the cutoff.

ically interested in the theory where the quadratically divergent terms are subtracted.

We have just seen that the bare potential (before and after the subtraction of the quadratic divergences) is everywhere stable. How can the renormalised potential show an instability? To answer this question, let us consider again Eq.(15) after the subtraction of the quadratically divergent terms.

As we already know, the instability occurs because the quantum fluctuations due to the fermions can compensate and then overwhelm the classical ϕ^4 term. Therefore, with no loss of generality, we can now neglect the bosonic contribution, as well as other unimportant finite terms, and limit ourselves to write:

$$V^{1l}(\phi) = \frac{1}{2}m_\Lambda^2\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 + \frac{g_\Lambda^4\phi^4}{16\pi^2}\ln\frac{\Lambda^2}{g_\Lambda^2\phi^2}. \quad (20)$$

At a lower scale μ ($< \Lambda$) we have:

$$V^{1l}(\phi) = \frac{1}{2}m_\mu^2\phi^2 + \frac{\lambda_\mu}{24}\phi^4 + \frac{g_\mu^4\phi^4}{16\pi^2}\ln\frac{\mu^2}{g_\mu^2\phi^2}, \quad (21)$$

which is the same potential of Eq.(20) written in terms of the renormalised parameters m_μ^2 , λ_μ and g_μ .

Clearly, if Eq.(20) does not show any instability, the same is true for Eq.(21). However, let us pretend (for the moment) that we have not made this observation and move to consider the usual phenomenological application of Eq.(21), which could have been obtained (section 3) within the \overline{MS} scheme.

From the first two terms of Eq.(21), the classical vacuum,

$$v^2 = -\frac{6m_\mu^2}{\lambda_\mu}, \quad (22)$$

is obtained. The last term can destabilise this vacuum if it becomes too large and negative. Strictly speaking, the presence of the last term also modifies the position of the classical minimum, but this is not a complication. In fact, although this is not a necessary step, we can slightly modify the above expressions by adopting renormalization conditions that keep the position of the minimum unchanged. With this choice, Eq.(21) is replaced by (see Appendix A):

$$V^{\text{I}}(\phi) = \frac{1}{2}m_v^2\phi^2 + \frac{\lambda_v}{24}\phi^4 - \frac{g_v^4\phi^4}{16\pi^2} \left(\ln \frac{\phi^2}{v^2} - \frac{3}{2} \right) - \frac{g_v^4 v^2}{8\pi^2} \phi^2. \quad (23)$$

In Eq.(23) we have defined the parameters m_v^2 , λ_v and g_v at the IR scale v , the classical (and quantum) minimum, which is now given by:

$$v^2 = -\frac{6}{\lambda_v} m_v^2. \quad (24)$$

Correspondingly, Eq.(20) is replaced by:

$$V^{\text{I}}(\phi) = \frac{1}{2}m_\Lambda^2\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 - \frac{g_\Lambda^4\phi^4}{16\pi^2} \left(\ln \frac{\phi^2}{\Lambda^2} - \frac{3}{2} \right) - \frac{g_\Lambda^4 v^2}{8\pi^2} \phi^2. \quad (25)$$

Going back to Eq.(23), we now look for values of λ_v , g_v and ϕ such that this equation is (expected to be) valid and, at the same time, give:

$$V^{\text{I}}(\phi) < V^{\text{I}}(v). \quad (26)$$

The usual requirements for the validity Eq.(23) are that the renormalised coupling constants, λ_v and g_v , as well as the quantum correction, $\frac{g_v^4}{16\pi^2} \ln \frac{\phi^2}{v^2}$, be perturbative, i.e.:

$$\lambda_v < 1 \quad , \quad g_v < 1, \quad (27)$$

and

$$\left| \frac{g_v^4}{16\pi^2} \ln \frac{\phi^2}{v^2} \right| < 1 \quad (28)$$

(note that Eqs.(27) and (28) are nothing but the perturbative conditions of Eq.(7) adapted to our current choices).

In the following we show that, contrary to the common expectation, Eqs.(27) and (28) are not sufficient to guarantee that Eq.(23) can be trusted. An additional condition has to be considered. As we shall see, the apparent instability of the potential is due to the neglect of this condition.

Let us choose λ_v and g_v so that these couplings, in addition to Eq.(27), also satisfy the relation:

$$\lambda_v = \frac{3g_v^4}{4\pi^2}. \quad (29)$$

Moreover, let us consider $\bar{\phi}$ such that:

$$\ln \frac{\bar{\phi}^2}{v^2} = 2. \quad (30)$$

Being $g_v < 1$, it is a trivial exercise to see that, by virtue of Eq.(30), Eq.(28) holds for $\bar{\phi}$. Moreover, inserting Eqs.(29) and (30) in Eq.(23), we find:

$$V^{1l}(\bar{\phi}) < V^{1l}(v). \quad (31)$$

We would conclude that, in the range of ϕ given by $v < \phi < \bar{\phi}$, the renormalised potential of Eq.(23) can be trusted and its instability (see fig.4) is theoretically well established. In fact, this is what is usually stated [8]. In order to avoid any misunderstanding, it is worth to stress that the RG-improvement cannot change this conclusion. In the range of ϕ that we consider here, the condition (28) holds so that, in this region, V^{1l} and V_{RGI} are very close one to the other .

As solid as they can seem, however, the above conclusions are incorrect. To understand why, let us first simplify (without any loss of generality) the discussion by neglecting in the following the running of m^2 and g . From Eqs.(23) and (25) we have then:

$$\frac{\lambda_\Lambda}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2} \ln \frac{\Lambda^2}{\phi^2} = \frac{\lambda_v}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2} \ln \frac{v^2}{\phi^2}, \quad (32)$$

which immediately gives:

$$\lambda_\Lambda = \lambda_v - \frac{3g^4}{2\pi^2} \ln \frac{\Lambda^2}{v^2}. \quad (33)$$

Inserting now Eqs.(29) and (30) in Eq.(32), we find:

$$\frac{\lambda_\Lambda}{24} + \frac{g^4}{16\pi^2} \ln \frac{\Lambda^2}{\bar{\phi}^2} = \frac{\lambda_v}{24} + \frac{g^4}{16\pi^2} \ln \frac{v^2}{\bar{\phi}^2} < 0. \quad (34)$$

Naturally, for the theory to be defined, it is $\lambda_\Lambda > 0$. Therefore, in order for Eq.(34) to be valid, we should have:

$$\frac{\Lambda^2}{\bar{\phi}^2} \leq 1. \quad (35)$$

Eq.(35) shows that, contrary to our naive expectation, $\bar{\phi}$ lies beyond the range of validity of V^{1l} (V_{RGI}).

We now understand the origin of the apparent instability of the renormalised potential. If, in order to decide whether a certain value of ϕ belongs to the region where V^{1l} can be trusted, we only consider Eqs. (27) and (28), *we loose the information contained in the additional independent condition:*

$$\frac{\lambda_v}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2} \ln \frac{v^2}{\phi^2} > 0. \quad (36)$$

When, on the contrary, this condition is taken into account, the effective potential does not present any instability. In other words, the instability occurs in a region of ϕ 's where Eq. (23) for V^{1l} is no longer valid.

Naturally, these same conclusions could have been reached by looking at the problem the other way around. In fact, coming back to the observation that we have put aside before, we note that, due to the condition $\phi^2 < \Lambda^2$, the combination $\frac{\lambda_\Lambda}{24} + \frac{g^4}{16\pi^2} \ln \frac{\Lambda^2}{\phi^2}$ cannot be negative. Therefore, Eq.(34) cannot be fulfilled and no instability can occur.

The above longer discussion, however, is motivated by the common believe that, in order to ascertain the validity of the result for V^{1l} , Eqs. (27) and (28) are the *only conditions* to be verified. Actually, this is the reason why it is still believed that the instability of V^{1l} (and V_{RGI}) is a genuine effect due to the quantum corrections.

We can now deepen our analysis by noting that, as an elementary exercise shows, the point beyond the minimum where the effective potential ceases to be convex, i.e. the inflection point in the external region, ϕ_{inf} , is such that:

$$\phi_{inf} \geq \Lambda. \quad (37)$$

Eq.(37) is important for two reasons. On the one hand, it shows that the effective potential is convex wherever it is defined. On the other hand, it provides a criterium for the derivation of lower bounds on the scalar (Higgs) mass.

To better understand this last point, let us consider the usual approach, where a bound on the renormalised λ is obtained from (the equivalent of) Eq.(33). At first it is noted that the instability occurs if $V^{1l}(\phi_0) = V^{1l}(v)$ at a certain ϕ_0 and $V^{1l}(\phi) < V^{1l}(v)$ for $\phi > \phi_0$. Then it is shown that ϕ_0 (almost) corresponds to the value of the running scale where $\lambda(\mu)$ vanishes (see Eq.(1) and footnote 3). Finally, a vanishing λ_Λ is taken in Eq.(33) so

that the highest possible physical cutoff Λ , corresponding to a given value of the renormalised coupling λ_v , is derived.

Instead, our analysis suggests that the upper bound for the range of ϕ 's, which is also the highest self-consistent value for the physical cutoff, should be taken at the inflection point of Eq.(37), the value of ϕ where the potential ceases to be convex.

Although up to now we have considered a simple scalar-Yukawa model, it is clear that our results are completely general. In the next section we shall see how the above criterium can be exported into the SM to get lower bounds on the Higgs mass.

Before we move to this phenomenological application, however, it is worth to stress that in the usual approach the requirement of stability appears to be an extra phenomenological constraint to be possibly imposed on the theory; an unstable potential is considered as a legitimate one. In fact, the meta-stability scenario, clearly excluded by our analysis, is based on the possibility of having a second minimum of the potential lower than the EW vacuum. As we have seen, however, the stability of the effective potential is an intrinsic property of the theory. No place is left for an unstable or meta-stable potential.

6 Lower bounds on the Higgs mass

Let us consider now some important phenomenological implications of our findings for the SM. Clearly, the first thing to point out is that, contrary to common belief, the Higgs effective potential does not present any instability. As for the determination of the lower bounds on m_H , we have seen that the internal consistency of the theory requires that the physical cutoff has to be taken at the location of the inflection point of the potential (in the region beyond the minimum).

Implementing this criterium for the determination of the scale of new physics, lower bounds for the Higgs mass are found. Our results will be compared with those obtained with the help of the usual instability criterium.

The well known one-loop potential of the scalar sector of the SM reads [34]:

$$V^1(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{64\pi^2} \left[\left(m^2 + \frac{\lambda}{2}\phi^2 \right)^2 \left(\ln \left(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2} \right) - \frac{3}{2} \right) \right]$$

$$\begin{aligned}
& +3 \left(m^2 + \frac{\lambda}{6} \phi^2 \right)^2 \left(\ln \left(\frac{m^2 + \frac{\lambda}{6} \phi^2}{\mu^2} \right) - \frac{3}{2} \right) + 6 \frac{g_1^4}{16} \phi^4 \left(\ln \left(\frac{\frac{1}{4} g_1^2 \phi^2}{\mu^2} \right) - \frac{5}{6} \right) \\
& + 3 \frac{(g_1^2 + g_2^2)^2}{16} \phi^4 \left(\ln \left(\frac{\frac{1}{4} (g_1^2 + g_2^2) \phi^2}{\mu^2} \right) - \frac{5}{6} \right) - 12 g^4 \phi^4 \left(\ln \frac{g^2 \phi^2}{\mu^2} - \frac{3}{2} \right) \Big], \quad (38)
\end{aligned}$$

where g_1 and g_2 are the weak interaction coupling constants, while g is the top-Yukawa coupling.

To have a well defined comparison between our criterium and the usual one, we have chosen to follow the work of Casas, Espinosa, and Quirós [11, 12]. In particular, we have taken their boundary conditions for g_1 , g_2 , m_t , ... at the scale M_Z as well as their matching conditions for the determination of the physical Higgs and top mass (see Appendix B and [11, 12] for details).

The RG improved potential, V_{RGI} , is obtained following the same steps of section 3. Naturally, the appropriate beta functions to consider in the RG equations are now the SM ones. As in [11, 12], we have used the two-loops beta functions [8]. Note also that, differently from our simpler model, we now have three additional RG equations, namely for g_1 , g_2 and g_S (the strong coupling), and that no analytic solution for the running of the couplings can be found. Choosing $t = \frac{1}{2} \ln \frac{\phi^2}{\mu^2}$, we get:

$$V_{RGI}(\phi) = m^2(t) \frac{\phi^2(t)}{2} + \lambda_{eff}(t) \frac{\phi^4(t)}{24} + \Omega(t), \quad (39)$$

where $\Omega(t)$ is the scale dependent vacuum energy, $\phi(t) = \xi(t) \phi$, with $\xi(t) = \exp \left(- \int_0^t \gamma(t') dt' \right)$ and $\gamma(t)$ being the Higgs anomalous dimension, and $\lambda_{eff}(t)$ is given by:

$$\begin{aligned}
\lambda_{eff}(t) = \lambda + \frac{3}{8\pi^2} \Bigg[& 6 \frac{g_1^4}{16} \left(\ln \left(\frac{g_1^2}{4} \right) - \frac{5}{6} \right) - 12 g^4 \left(\ln g^2 - \frac{3}{2} \right) \\
& + 3 \frac{(g_1^2 + g_2^2)^2}{16} \left(\ln \left(\frac{g_1^2 + g_2^2}{4} \right) - \frac{5}{6} \right) \Bigg], \quad (40)
\end{aligned}$$

with $\lambda = \lambda(t)$, $g = g(t)$, $g_1 = g_1(t)$, $g_2 = g_2(t)$.

First, we have checked that, when the usual $V_{RGI} = 0$ criterium is used, meaning that the scale of new physics, Λ , is determined as the value of ϕ where [12]:

$$\lambda_{eff} + 12 \frac{m^2}{\xi^2 \Lambda^2} + 24 \frac{\Omega}{\xi^4 \Lambda^4} = 0, \quad (41)$$

Λ (TeV)	M_H^{inf} (Gev)	M_H (Gev)	ΔM_H (Gev)
1	66	55.5	10.5
5	88	81	7
10	94.5	88.5	6
100	108.5	105.5	3
1000	117	115	2
10^{16}	137.5	137.5	0

Table 1: Lower bounds on the Higgs mass as a function of the physical cutoff. The values of the physical parameters are chosen according to [11, 12] (see also Appendix B). The second and third columns contain the bounds obtained with the convexity and instability criterium respectively.

the results of [11, 12] are recovered. Then, we have derived the physical cutoff according to our criterium, i.e. we have looked for the location of the external inflection point of V_{RGI} .

In Table 1 we summarise the results obtained with these two criteria for different values of Λ . For small cutoffs, the lower bounds on M_H given by our criterium are ~ 10 Gev larger than the current determinations [12], while for increasing values of Λ the difference tends to disappear.

The convergence between these two methods (for large cutoffs) has a simple explanation. Let us neglect, for a moment, the convexity constraint. As M_H increases, the location of the inflection point moves to higher and higher values of ϕ . The same is, obviously, true for the point where the potential vanishes. In this region, V_{RGI} is very well approximated by Eq.(1) and $\bar{\lambda}(\phi)$ changes very slowly with ϕ . Therefore, the two criteria practically give one and the same value for Λ .

The scope of Table 1 is to provide a comparison between the two different methods for the determination of lower bounds on m_H . To this end, the values of the physical parameters have been chosen according to [11, 12] (see Appendix B) rather than to their more recent measured values. The reader can easily verify that the results we have found with the usual criterium (reported in the third column of Table 1) agree with those of [11, 12].

Now, considering the updated values: $M_Z = 91.2$ Gev, $M_W = 80.4$ Gev, $\alpha_s = 0.119$ [35] and $M_t = 178$ Gev [36], we find for the lower bounds on M_H the results reported in Table 2. Note that, taking into account the present

Λ (TeV)	M_H^{inf} (Gev)	M_H (Gev)	ΔM_H (Gev)
1	68.5	57.5	11
5	91.5	84	7.5
10	98	92	6
100	113	109.5	3.5
1000	122	120	2
10^{16}	143.5	143.5	0

Table 2: Lower bounds on the Higgs mass as a function of the physical cutoff. Differently from Table 1, the physical parameters have been chosen according to their most recent experimental determinations (see text). As for Table 1, the second and third columns contain the bounds obtained with the convexity and instability criterium respectively.

experimental uncertainty on M_t [36], $M_t = 178 \pm 4.3$, we get: $M_H = 68.5^{+3}_{-3.5}$ for $\Lambda = 1$ Tev up to $M_H = 143.5 \pm 8.5$ for $\Lambda = 10^{19}$ Gev.

7 Wilsonian RG

In the previous section we have considered a phenomenological application of our findings. Now, to further support our results, we come back to the simpler Higgs–Yukawa model of Eq.(4) and show that, with the help of the Wilsonian RG method, our analysis can be extended beyond perturbation theory.

For the Euclidean Wilsonian action of our model at the running scale k , $S_k[\phi, \bar{\psi}, \psi]$, we consider the following non-perturbative ansatz [37]:

$$S_k[\phi, \bar{\psi}, \psi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \bar{\psi} \gamma_\mu \partial_\mu \psi + U_k(\phi, \bar{\psi}, \psi) \right). \quad (42)$$

As for the case of the scalar theory (see section 2 and [23]), for the internal region we expect that from Eq.(42) a non-perturbative flow equation can be obtained which reproduces the Maxwell construction. Here, however, our scope is to investigate the possibility of having an instability of the scalar potential in the region beyond the minimum. Therefore, we only consider this region, where the running for the Wilsonian potential of our model is given by the non-perturbative RG equation [37]:

$$\begin{aligned} \frac{\partial U_k(\phi, \sigma)}{\partial k} = & -\frac{k^3}{16\pi^2} \ln \left(\frac{k^2 + U_k''(\phi, \sigma)}{k^2 + U_k''(0, \sigma)} \right) + \frac{k^3}{4\pi^2} \ln \left(1 + \frac{\dot{U}_k^2(\phi, \sigma)}{k^2} \right) \\ & - \frac{k^3}{16\pi^2} \ln \left(1 + \frac{2\sigma \dot{U}_k(\phi, \sigma)}{k^2 + \dot{U}_k^2(\phi, \sigma)} \left(\ddot{U}_k(\phi, \sigma) - \frac{\dot{U}_k'^2(\phi, \sigma)}{k^2 + U_k''(\phi, \sigma)} \right) \right). \end{aligned} \quad (43)$$

Here $\sigma = \bar{\psi}\psi$, the prime indicates the derivative w.r.t. ϕ and the dot the derivative w.r.t. σ .

The bare value of the potential, which is nothing but the boundary condition for the RG equation (43), is (see Eq.(4)):

$$U_\Lambda(\phi, \sigma) = \frac{1}{2}m_\Lambda^2\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 + g_\Lambda\phi\sigma. \quad (44)$$

We now consider for $U_k(\phi, \sigma)$ the additional truncation:

$$U_k(\phi, \sigma) = V_k(\phi) + G_k(\phi)\sigma, \quad (45)$$

which means that we neglect the contributions from higher powers of $\bar{\psi}\psi$.

Inserting Eq.(45) in Eq.(43), we finally get the RG equations:

$$\begin{aligned} \frac{\partial V_k(\phi)}{\partial k} &= -\frac{k^3}{16\pi^2} \ln \left(\frac{k^2 + V_k''(\phi)}{k^2 + V_k''(0)} \right) + \frac{k^3}{4\pi^2} \ln \left(1 + \frac{G_k^2(\phi)}{k^2} \right) \\ \frac{\partial G_k(\phi)}{\partial k} &= -\frac{k^3}{16\pi^2} \frac{1}{k^2 + V_k''(\phi)} \left(G_k''(\phi) - \frac{2G_k(\phi)G_k'(\phi)}{k^2 + G_k^2(\phi)} \right). \end{aligned} \quad (46)$$

From Eq.(44) is clear that the boundary conditions for V_k and G_k are:

$$\begin{aligned} V_\Lambda(\phi) &= \frac{1}{2}m_\Lambda^2\phi^2 + \frac{1}{24}\lambda_\Lambda\phi^4 \\ G_\Lambda(\phi) &= g_\Lambda\phi. \end{aligned} \quad (47)$$

Given m_Λ^2 , λ_Λ and g_Λ at $k = \Lambda$, we can run the RG equations (46) to get for the scalar effective potential, $V_{eff}(\phi)$, the non-perturbative approximation: $V_{wil}(\phi) = V_{k=0}(\phi)$. Choosing: $\lambda_\Lambda = 5 \cdot 10^{-2}$, $m_\Lambda^2 = -1 \cdot 10^{-2}$, $g_\Lambda = 5 \cdot 10^{-1}$ at $\Lambda = 100$, i.e. taking the same values used in fig.6, we get for V_{wil} the result plotted in fig.8 (we remind that the RG equation (43) is valid only in the external region).

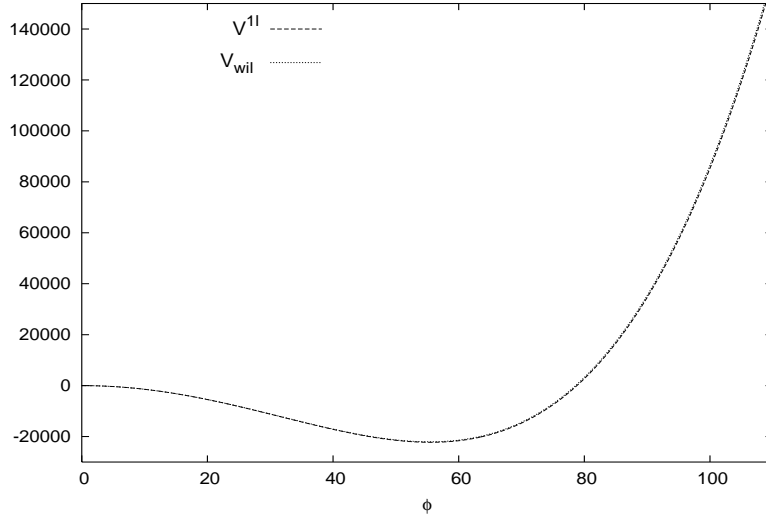


Figure 8: The Wilsonian, $V_{wil} = V_{k=0}$, effective potential. The boundary values of the parameters are as in Fig.6. Only the region external to the minimum has to be considered. For comparison we have also plotted the one-loop effective potential of Fig.6. We see that, as explained in the text, V_{wil} and V^{1l} are very close one to the other.

For comparison, we have also plotted the corresponding V^{1l} (which is nothing but the potential of fig.6). As we can easily see, V_{wil} and V^{1l} are very close one to the other. This result could have been guessed. As we have already said, in fact, in the external region the path integral that defines the effective potential is dominated by a single saddle point. As a consequence, we expect that the loop-expansion, and in particular the one-loop potential, provides a good approximation for V_{eff} . The close coincidence between V^{1l} (perturbative) and V_{wil} (non-perturbative) supports this expectation.

By its own construction, the Wilsonian method does not contain any ad hoc subtraction of terms. This is why we have compared the effective potential found with Eqs.(46) with the original one-loop result, the potential of fig.6, where the quadratically divergent terms were kept.

If we want to make contact with the perturbative V^{1l} where the quadratic divergences are subtracted (fig.7), we need to implement this operation in the flow equations.

Performing a polynomial expansion of $V_k(\phi)$ and $G_k(\phi)$, we easily see that

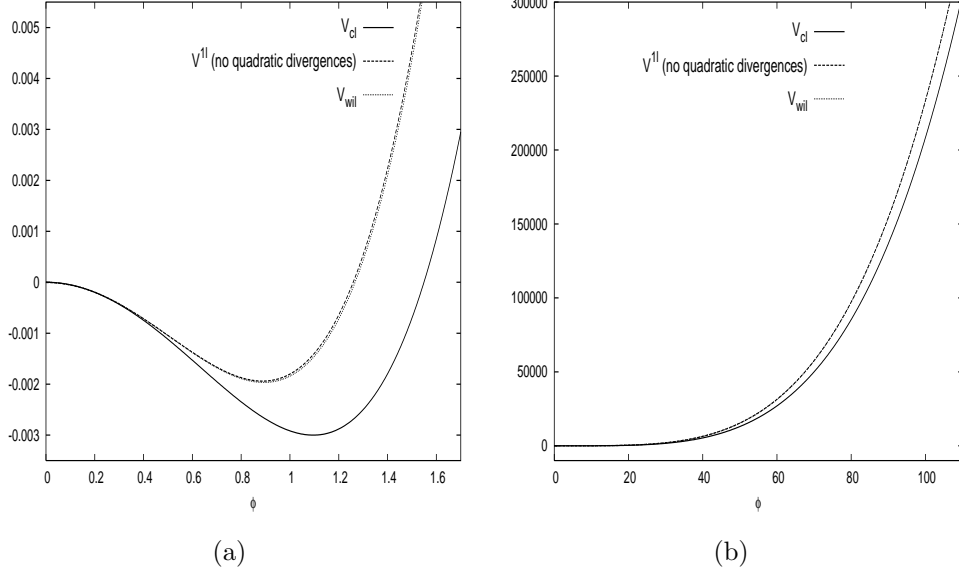


Figure 9: The bare together with the Wilsonian potential after subtraction of the quadratic divergences. The boundary values of the parameters are as in Fig.6. For V_{wil} , only the external region has to be considered. For comparison, we have also plotted the one-loop effective potential of Fig.7. We see that, even after subtracting the quadratic divergences, V_{wil} and V^{1l} are quite close.

the subtraction of the quadratic divergences in our flow equations amounts to add the term:

$$\left(-\frac{\lambda_k}{32\pi^2} + \frac{g_k^2}{4\pi^2} \right) k \phi^2 \quad (48)$$

to the first of Eqs.(46). In Eq.(48), λ_k is the coefficient of ϕ^4 in the expansion of $V_k(\phi)$, while g_k is the coefficient of ϕ in the expansion of $G_k(\phi)$. Moreover, at each step of the RG iteration, λ_k and g_k are determined via a polynomial fit of V_k and G_k respectively. Their boundary values, of course, are λ_Λ and g_Λ .

Taking for m_Λ^2 , λ_Λ , g_Λ and Λ the same values considered above, we now run the modified system of RG equations and get for V_{wil} the result plotted

in figs.9. As before, we note that V_{wil} and V^{1l} are very close.

The results of the present section strongly support our previous findings. Even within the non-perturbative framework considered here, the effective potential does not show any sign of instability.

8 Summary and conclusions

Starting with the analysis of some popular, but misleading, arguments, we have studied the instability problem of the EW vacuum with the help of a Higgs–Yukawa model.

Combining the Bogolubov approach to symmetry breaking, namely the criterium of dynamical instability, with the Wilsonian RG method, we have shown that there is no conflict between the convexity of the effective potential (effective action) and the existence of broken phase vertex functions. This preliminary step was helpful in establishing the incorrectness of the above quoted arguments.

Successively, we have shown that the potential instability is due to an illegal extrapolation of the renormalised effective potential into a region where the results of renormalised perturbation theory do not hold. Moreover, in agreement with what is expected from general theorems, we have found that the effective potential of the cutoff Higgs–Yukawa model is convex allover the region where is defined.

To establish these results, it was necessary to go beyond the usual application of the perturbation theory conditions. In this respect, we note that the dimensional regularization scheme, by its own construction, directly gives the results of renormalised perturbation theory. As the subject of this paper shows, however, the connection between the UV and the IR sector of the theory (the relation between bare and renormalised theory) can present aspects which are hidden to a naive application of dimensional regularization.

In our case, the consistency constraint for the theory ($\phi \leq \Lambda$) and Eq.(32) imply that the combination $\frac{\lambda_v}{24} + \frac{g^4}{16\pi^2} \ln \frac{v^2}{\phi^2}$ cannot be negative. When we blindly jump to the perturbation theory results, this information is lost. Actually, Eqs.(27) and (28), typically considered as the only conditions for the renormalised perturbation theory to hold, do not contain the above independent constraint. The effective potential appears to be unstable when this condition is ignored.

We started our analysis within the framework of the momentum cut-

off regularization scheme. Successively, with the help of the Wilsonian RG method, our results were established in a more general non-perturbative context.

Moreover, despite the stability of the potential, we have shown that lower bounds on the Higgs mass can still be derived. In fact, for a given renormalised value of λ , the corresponding cutoff can be found looking for the inflection point of V_{eff} in the external region ($\phi > v$). If the scale of new physics is not too high, a sizeable difference between our bounds and the usual ones is obtained. For Λ in the TeV region, we find a value of m_H which is some 10 – 11 GeV higher than the current determination.

In addition to these phenomenological applications, it is worth to note that there is a deep conceptual difference between our analysis and the usual one. While in our case the stability of the potential, as well as the bounds on m_H , come as a manifestation of the internal consistency of the theory, in the usual approach the bounds are the result of an (apparently) additional constraint to be imposed on the potential, the requirement of stability. The instability is considered as a theoretically legitimate possibility. In fact, the meta-stability scenario explores the consequences of having a minimum lower than the EW one. Our results exclude this scenario.

In the present work we have been interested on the instability issue only. However, we believe that our results come as a manifestation of a general problem, the (somehow delicate) connection between the UV and the IR sector of a theory and that a similar analysis can be applied to other cases where this connection is expected to play an important role. We hope to come to this point in the future.

As already said, we come now to the comparison of our work with [26, 27]. First of all we note that the instability problem concerns the renormalised effective potential. Therefore, it is important to perform the analysis within a range of ϕ where renormalised perturbation theory is (or is supposed to be) valid. In [26, 27], however, the potential has a minimum at the cutoff, i.e. at $\phi \sim \Lambda = \frac{\pi}{a}$ (see fig.2 of [26] and fig.4 of [27]) and all the relevant scales, namely the “low energy scale” μ , the cutoff scale Λ and the minimum v are of the same order. In our opinion, this hardly helps in understanding the origin of the instability problem.

Moreover, the renormalised potential (see Eq.(2) in [27]) is obtained from the (subtracted) bare potential (see Eq.(5) in [27]) after expanding in $\frac{\phi}{\Lambda}$ and neglecting negative powers of Λ . Insisting on the difference between the bare and the renormalised potential for values of ϕ beyond Λ , as done by the

authors, once more does not help in clarifying the problem.

We believe that we have clearly identified the origin of the apparent instability of the effective potential. Contrary to what is stated in [27], it seems to us that it has nothing to do with the triviality of the theory.

Finally, we note that in [26, 27], in order to avoid problems with the convexity of V_{eff} (as stated by the authors), the constrained potential is used. On the contrary, insisting on the convexity of V_{eff} as a guiding property, we have found the flaw that artificially makes V_{eff} unstable in the external region.

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A Renormalization conditions

In this Appendix we compute the renormalised potential of Eq.(23), where the renormalization conditions that keep the minimum and the curvature around the minimum fixed at their classical values are implemented. Clearly, these conditions are:

$$\left(\frac{dV^{1l}}{d\phi}\right)_{\phi=v} = 0 \quad (49)$$

$$\left(\frac{d^2V^{1l}}{d\phi^2}\right)_{\phi=v} = \frac{\lambda v^2}{3} = -2m^2, \quad (50)$$

with $v = \sqrt{\frac{-6m^2}{\lambda}}$. From Eq.(15) we get:

$$\begin{aligned} \frac{dV^{1l}}{d\phi} &= \phi \left(m^2 + \delta m^2 + (\lambda + \delta\lambda) \frac{\phi^2}{6} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2} \right) \Lambda^2 \right. \\ &\quad \left. + \frac{\lambda}{32\pi^2} \left(m^2 + \frac{\lambda}{2} \phi^2 \right) \ln \frac{m^2 + \frac{\lambda}{2} \phi^2}{\Lambda^2} - \frac{g^4 \phi^2}{4\pi^2} \ln \frac{g^2 \phi^2}{\Lambda^2} \right), \end{aligned} \quad (51)$$

so that the condition (49) becomes:

$$\begin{aligned} 0 &= \delta m^2 + \delta\lambda \frac{v^2}{6} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2} \right) \Lambda^2 \\ &\quad + \frac{\lambda}{32\pi^2} \left(m^2 + \frac{\lambda}{2} v^2 \right) \ln \frac{m^2 + \frac{\lambda}{2} v^2}{\Lambda^2} - \frac{g^4 v^2}{4\pi^2} \ln \frac{g^2 v^2}{\Lambda^2}. \end{aligned} \quad (52)$$

Deriving V^{1l} once more w.r.t. ϕ , we get:

$$\begin{aligned} \frac{d^2V^{1l}}{d\phi^2} &= m^2 + \delta m^2 + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2} \right) \Lambda^2 + \frac{\lambda}{32\pi^2} \left(m^2 + \frac{3}{2} \lambda \phi^2 \right) \ln \frac{m^2 + \frac{\lambda}{2} \phi^2}{\Lambda^2} \\ &\quad + \frac{\phi^2}{2} \left(\lambda + \delta\lambda + \frac{\lambda}{16\pi^2} - \frac{3g^4}{2\pi^2} \left(\ln \frac{g^2 \phi^2}{\Lambda^2} + \frac{2}{3} \right) \right), \end{aligned} \quad (53)$$

and the condition (50) reads:

$$\begin{aligned}
0 &= \delta m^2 + \delta \lambda \frac{v^2}{2} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2} \right) \Lambda^2 + \frac{\lambda}{32\pi^2} \left(m^2 + \frac{3}{2} \lambda v^2 \right) \ln \frac{m^2 + \frac{\lambda}{2} v^2}{\Lambda^2} \\
&+ \frac{v^2}{2} \left(\frac{\lambda^2}{16\pi^2} - \frac{3g^4}{2\pi^2} \left(\ln \frac{g^2 v^2}{\Lambda^2} + \frac{2}{3} \right) \right). \quad (54)
\end{aligned}$$

From Eqs.(52) and (54) we find:

$$\delta \lambda = \frac{3g^4}{2\pi^2} \left(\ln \frac{g^2 v^2}{\Lambda^2} + 1 \right) - \frac{3\lambda^2}{32\pi^2} \left(\ln \frac{m^2 + \frac{\lambda}{2} v^2}{\Lambda^2} + 1 \right) \quad (55)$$

$$\delta m^2 = \left(\frac{g^2}{4\pi^2} - \frac{\lambda}{32\pi^2} \right) \Lambda^2 - \frac{\lambda m^2}{32\pi^2} \left(\ln \frac{m^2 + \frac{\lambda}{2} v^2}{\Lambda^2} + 3 \right) - \frac{g^4 v^2}{4\pi^2}. \quad (56)$$

Inserting Eqs.(55) and (56) in V^{1l} , i.e. in Eq.(15), we finally get:

$$\begin{aligned}
V^{1l} &= \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 + \frac{(m^2 + \frac{\lambda}{2} \phi^2)^2}{64\pi^2} \left(\ln \frac{m^2 + \frac{\lambda}{2} \phi^2}{m^2 + \frac{\lambda}{2} v^2} - \frac{3}{2} \right) \\
&- \frac{g^4 \phi^4}{16\pi^2} \left(\ln \frac{\phi^2}{v^2} - \frac{3}{2} \right) + \frac{v^2}{2} \phi^2 \left(\frac{3\lambda^2}{32\pi^2} - \frac{g^4}{4\pi^2} \right). \quad (57)
\end{aligned}$$

Finally, neglecting the bosonic contribution to the quantum fluctuation determinant, we see that Eq.(57) is nothing but the renormalised one-loop potential of Eq.(23).

B RG-improved Potential for the SM

In the present Appendix we provide some useful relations needed for the computation of the RG-improved one-loop effective potential of the SM (section 6). Following [11], the matching conditions for the Higgs and the top masses are taken as:

$$M_H^2(t) = m_H^2(t^*) \frac{\xi^2(t^*)}{\xi^2(t)} + \text{Re} \left(\Pi(p^2 = M_H^2) - \Pi(p^2 = 0) \right) \quad (58)$$

$$M_t = m_t(M_t) \left(1 + \frac{g_s(M_t)^2}{3\pi^2} \right), \quad (59)$$

where Π is the self-energy of the Higgs boson (for the full explicit expression see the appendix A of [11]). Moreover, although the exact effective potential is scale independent, for V^{1l} and V_{RGI} this is true only approximately. The value t^* of the parameter t that appears in Eq.(58) is chosen as to minimize the dependence of V_{RGI} on the choice of the running scale $\mu(t) = M_Z e^t$. The corresponding $\mu(t^*)$, in our case, is: $\mu(t^*) \sim 130$ Gev.

Accordingly, omitting the Higgs and the Goldstone (negligible) contributions, the value of $m_H^2(t^*)$ is secured as [11]:

$$\begin{aligned}
m_H^2(t^*) = & \xi^2(t^*)v^2 \left(\frac{\lambda(t^*)}{3} + \frac{3}{64\pi^2} \left\{ g_1^4(t^*) \left[\log \frac{g_1^2(t^*)\xi^2(t^*)v^2}{4\mu^2(t^*)} + \frac{2}{3} \right] \right. \right. \\
& + \frac{1}{2} [g_1^2(t^*) + g_2^2(t^*)]^2 \left[\log \frac{[g_1^2(t^*) + g_2^2(t^*)]\xi^2(t^*)v^2}{4\mu^2(t^*)} + \frac{2}{3} \right] \\
& \left. \left. - 8g^4(t^*) \log \frac{g^2(t^*)\xi^2(t^*)v^2}{2\mu^2(t^*)} \right\} \right), \tag{60}
\end{aligned}$$

where in the first term of the r.h.s. we recognise the tree-level relation for m_H^2 , while the other terms come from the loop corrections.

The boundary values for the coupling constants are choosen as [11]:

$$\begin{aligned}
g_1(M_Z) &= 0.650 \\
g_2(M_Z) &= 0.355 \\
g_s(M_Z) &= 1.218 \\
\gamma(M_Z) &= 0 \\
\Omega(M_Z) &= 0 \\
g(M_t) &= \frac{\sqrt{2}m_t(M_t)}{\xi(M_t)v} = 0.9635, \tag{61}
\end{aligned}$$

which correspond to $M_W = 80$ Gev, $M_Z = 91.2$ Gev, $\alpha_s = 0.118$ and $M_t = 175$ Gev.

The coupling $\lambda(M_Z)$ is kept as a free parameter. As explained in the text, by considering different values of $\lambda(M_Z)$, we obtain different values for the physical cutoff, thus getting lower bounds for the Higgs mass as a function of the scale of new physics.

Note also that, in order to keep the location of the minimum to its phenomenological value, m^2 has to be fixed by the condition: $\frac{\langle \phi(t^*) \rangle}{\xi(t^*)} = v = 246.22$ Gev, which gives [11]:

$$\begin{aligned}
m^2(t^*) &= -\xi^2(t^*)v^2 \left(\frac{\lambda(t^*)}{6} + \frac{3}{64\pi^2} \left\{ \frac{1}{2}g_1^4(t^*) \left[\log \frac{g_1^2(t^*)\xi^2(t^*)v^2}{4\mu^2(t^*)} - \frac{1}{3} \right] \right. \right. \\
&+ \frac{1}{4} [g_1^2(t^*) + g_2^2(t^*)]^2 \left[\log \frac{[g_1^2(t^*) + g_2^2(t^*)]\xi^2(t^*)v^2}{4\mu^2(t^*)} - \frac{1}{3} \right] \\
&\left. \left. - 4g^4(t^*) \left[\log \frac{g^2(t^*)\xi^2(t^*)v^2}{2\mu^2(t^*)} - 1 \right] \right\} \right). \tag{62}
\end{aligned}$$

Now, solving numerically the system of RG equations for the running coupling constants, we get Eq. (39) of section 6 for $V_{RGI}(\phi)$.

We end this appendix giving the boundary values of the coupling constants corresponding to the updated values of M_Z , M_W , α_s and M_t reported in section 6: $g_1(M_Z) = 0.653$, $g_2(M_Z) = 0.349$, $g_s(M_Z) = 1.223$ and $g(M_t) = 0.980$.